Numerical Recipes in C++

The Art of Scientific Computing

Second Edition

William H. Press

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Preface to the C++ Edition

C++ has gradually become the dominant language for computer programming, displacing C and Fortran even in many scientific and engineering applications. This version of *Numerical Recipes* contains the entire text of the Second Edition with all the programs presented in C++.

C++ poses special problems for numerical work. In particular, it is difficult to treat vectors and matrices in a manner that is simultaneously efficient and yet allows programming with high-level constructs. The fact that there is still no universally accepted standard library for doing this makes the problem even more difficult for authors of a book like this one. In Chapter 1 and the Appendices we describe how we have solved this problem. The default option is for you, the reader, to use a very simple class library that we provide. You can be up and running in a few minutes. We also show you how you can alternatively use any other matrix/vector class library of your choosing. This may take you a few minutes to set up the first time, but thereafter will provide transparent access to the Recipes with essentially no loss in efficiency.

We have taken this opportunity to respond to a clear consensus from our C readers, and converted all arrays and matrices to be "zero-based." We have also taken this opportunity to fix errors in the text and programs that have been reported to us by our readers. There are too many people to acknowledge individually, but to all who have written to us we are very grateful.

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Brian P. Flannery

This is the revised and expanded second edition of the hugely popular Numerical Recipes: the Art of Scientific Computing. The product of a unique collaboration among four leading scientists in academic research and industry, Numerical Recipes is a complete text and reference book on scientific computing. About the Authors. N/A. Reviews and Rating: Amazon. Related Book Categories: Numerical Analysis and Scientific Computing. NUMERICAL RECIPES. The Art of Scientific Computing. Third Edition. William H. Press. Raymer Chair in Computer Sciences and Integrative Biology The University of Texas at Austin. Saul A. Teukolsky. Hans A. Bethe Professor of Physics and Astrophysics Cornell University.Â To the extent that this edition of Numerical Recipes in C has a more graceful and â€œC-likeâ€ programming style than its predecessor, most of the credit goes to Seth. (Of course, we accept the blame for the Fortranish lapses that still remain.) We prepared this book for publication on DEC and Sun workstations running the UNIX operating system and on a 486/33 PC compatible running MS-DOS 5.0 / Windows 3.0.