

A Course in Model Theory I:

Introduction¹

Rami Grossberg

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

¹This **preliminary draft** is dated from January 17, 2015. The book will be published by Cambridge University Press. The book is approximately 93.82% complete, I expect the final version to have about 745 pages, many sections of the current version will be revised and few will be added. I hope to have a stable version of this volume soon.

This version is made only for students studying model theory with me and not for distribution outside CMU. If you have a copy not received directly from me, it is an illegal copy and I request that you will not share with others.

Exercise #=701.

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Finite model theory arose as an independent field of logic from consideration of problems in theoretical computer science. Basic concepts in this field are finite graphs, databases, computations etc. One of the underlying observations behind the interest in finite model theory is that many of the problems of complexity theory and database theory can be formulated as problems of mathematical logic, provided that we limit ourselves to finite structures. While the objects of study in finite model theory are finite structures, it is often possible to make use of infinite structures in the proofs. W