The Korteweg-de Vries Equation: History, exact Solutions, and graphical Representation

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Travelling waves as solutions to the Korteweg-de Vries equation (KdV) which is a non-linear Partial Differential Equation (PDE) of third order have been of some interest already since 150 years.
The author's aim is to present an analytical exact result to the KdV equation by means of elementary operations as well as by using Bäcklund transform. Special interest is devoted to non-linear superposition of several waves, the so called Solitary Waves or Solitons which is performed by Bäcklund transform too.
The derivation of the exact solutions follows the presentation of Vvedenski (1992).
It is fascinating that these exact solutions which involve a lot of calculations can be done with the help of the Computer Algebra System *Mathematica* © (see Wolfram (1999), not only presenting analytical expressions but in addition 3D-Plots, Contour Plots, and 2D-Plots for discrete values of time, finally animating these 2D-Plots and presenting these plots as well as the solution formulae at an Internet page.
Another aim of the author has been to apply numerical discrete methods to non-linear PDEs. With the help of the exact solutions gained here an excellent possibility is given to test the quality of the numerical methods by checking the numerical results against the exact solution.

§ 1: Historical Background

A nice story about the history and the underlying physical properperties of the Korteweg-de Vries equation can be found at an Internet page of the Herriot-Watt University in Edinburgh (Scotland), see Eilbeck (1998). The following text is taken from that page:
Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. Here his original text as he described it in Russell (1845):

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour,

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preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

This event took place on the Union Canal at Hermiston, very close to the Riccarton campus of Heriot-Watt University, Edinburgh.
Throughout his life Russell remained convinced that his solitary wave (the ‘‘Wave of Translation”) was of fundamental importance, but ninteenth and early twentieth century scientists thought otherwise. His fame has rested on other achievements. To mention some of his many and varied activities, he developed the "wave line" system of hull construction which revolutionized ninteenth century naval architecture, and was awarded the gold medal of the Royal Society of Edinburgh in 1837. He began steam carriage service between Glasgow and Paisley in 1834, and made the first experimental observation of the "Doppler shift" of sound frequency as a train passes. He reorganized the Royal Society of Arts, founded the Institution of Naval Architects and in 1849 was elected Fellow of the Royal Society of London.
He designed (with Brunel) the "Great Eastern" and built it; he designed the Vienna Rotunda and helped to design Britain's first armoured warship (the "Warrior"). He developed a curriculum for technical education in Britain, and it has recently become known that he attempted to negotiate peace during the American Civil War.

It was not until the mid 1960's when applied scientists began to use modern digital computers to study nonlinear wave propagation that the soundness of Russell's early ideas began to be appreciated. He viewed the solitary wave as a self-sufficient dynamic entity, a "thing" displaying many properties of a particle. From the modern perspective it is used as a constructive element to formulate the complex dynamical behaviour of wave systems throughout science: from hydrodynamics to nonlinear optics, from plasmas to shock waves, from tornados to the Great Red Spot of Jupiter, from the elementary particles of matter to the elementary particles of thought.
For a more detailed and technical account of the solitary wave, see Bullough (1988).

The phenomenon described by Russell can be expressed by a non-linear Partial Differential Equation of the third order. To remind: A partial differential equation (PDE) is a mathematical equation which contains an unknown function of more than one variable as well as some derivatives of that function with respect to the different independent variables. In practical applications where the PDE describes a dynamic process one of the variables has the meaning of the time (hence denoted by t) and the other (normally only up to 3) variable have the meaning of the space (hence denoted by x, y and z)
We consider the simplest case in only one space variable x. So we are looking for a function u depending on the variables x and t, i.e. $u(x,t)$ which describes the elongation of the wave at the place x at time t.

Using the typical short denotations as

$$u_t(x,t) = \frac{\partial u(x,t)}{\partial t}; \quad u_x(x,t) = \frac{\partial u(x,t)}{\partial x}; \quad u_{xx}(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}; \quad u_{xxx}(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3}$$

the problem can be formulated as:

(1) $$u_t(x,t) + 6\,u(x,t)\,u_x(x,t) + u_{xxx}(x,t) = 0$$

This is the Korteweg-de Vries Equation (KdV) which is nonlinear because of the product shown in the second summand and which is of third order because of the third derivative as highest in the third summand. The factor 6 is just a scaling factor to make solutions easier to describe.

The aim here is to find general exact solutions to (1), i.e.: here we have neither initial conditions nor boundary conditions. The solutions to (1) are called *Solitons* or *Solitary waves*.

The Korteweg-de Vries is a hyperbolic PDE in the general sense of the hyperbolicity definition. From that it follows that it describes a reversible dynamical process.

The author's interest for analytical solutions of (1) stems from the fact that in applying numerical methods to non-linear PDEs, the KdV equation is well suited as a test object, since having an analytical solution statements can be made on the quality of the numerical solution in comparing the numerical result to the exact result.

The main part of the subsequent deduction of an analytical solution to (1) is taken from Vvedenskii (1992), who has shown how to find an exact solution to the KdV equation and who has used the tool of Bäcklund transform to obtain an analytical solution and who - in addition to that - performs non-linear superposition of two, three and more solutions corresponding to two, three or more soliton waves by using Bäcklund transform again.

§ 2: Exact Solution to the KdV Equation

We remember that the simplest mathematical wave is a function of the form $u(x,t) = f(x - c\,t)$ which e.g. is a solution to the simple PDE $u_t + c\,u_x = 0$ where c denotes the speed of the wave. For the well known wave equation $u_{tt} - c^2\,u_{xx} = 0$ the famous d'Alembert solution leads to two wave fronts represented by terms $f(x - c\,t)$ and $f(x + c\,t)$.
Hence we start here with a trial solution

(2) $$u(x,t) = z(x - \beta\,t) \equiv z(\xi)$$

just denoting the parameter c above by β here and the function f by z.
Substituting the trial solution (2) into (1) we are led to the Ordinary Differential Equation

(3) $$-\beta\,\frac{dz}{d\xi} + 6\,z\,\frac{dz}{d\xi} + \frac{d^3z}{d\xi^3} = 0$$

Integration can be done directly since (3) is a form of a total derivative. It follows from (3):

(4) $$-\beta z + 3z^2 + \frac{d^2 z}{d\xi^2} = c_1$$

where c_1 is the constant of integration. In order to obtain a first order equation for z a multiplication with $\frac{dz}{d\xi}$ is done, i.e.:

$$-\beta z \frac{dz}{d\xi} + 3z^2 \frac{dz}{d\xi} + \frac{d^2 z}{d\xi^2} \cdot \frac{dz}{d\xi} = c_1 \frac{dz}{d\xi}$$

$$\Rightarrow \quad -\beta z\, dz + 3z^2\, dz + \frac{d^2 z}{d\xi^2}\, dz = c_1\, dz$$

Integration on both sides (with c_2 as the constant of integration) leads to

(5) $$-\frac{\beta}{2} z^2 + z^3 + \frac{1}{2}\left(\frac{dz}{d\xi}\right)^2 = c_1 z + c_2$$

Now it is required that in case $x \to \pm\infty$ we should have $z \to 0$, $\frac{dz}{d\xi} \to 0$, $\frac{d^2 z}{d\xi^2} \to 0$.
From these requirements it follows $c_1 = c_2 = 0$.

Remark:
More general solutions can be found for other choices of c_1 and c_2. These solutions can be represented in terms of elliptic integrals, for details see Drazin (1983).
With $c_1 = c_2 = 0$ equation (5) can be written as

(6) $$\left(\frac{dz}{d\xi}\right)^2 = z^2(\beta - 2z)$$

By seperation of variables we may write

(7) $$\int_0^z \frac{d\zeta}{\zeta\sqrt{\beta - 2\zeta}} = \int_0^\xi d\eta$$

The choice of 0 for the lower integration limits does not bring any loss of generality since the starting point can be transformed linearily.
The integration of the left hand side of (7) can be done by using a transformation

(8) $$s := \tfrac{1}{2}\beta\, sech^2 w$$

The role of *s* here is played by the variable ζ and we obtain

(9) $$\beta - 2\zeta = \beta(1 - sech^2 w) = \beta \tanh^2 w$$

since the relation $\cosh^2 w - \sinh^2 w = 1$ holds. Furthermore we have

(10) $$\frac{d\zeta}{dw} = -\beta \frac{\sinh w}{\cosh^3 w}$$

The upper integration limit of the left hand integral in (7) due to (8) is transformed to

(11) $$w = sech^{-1}\sqrt{\frac{2z}{\beta}}$$

Substituting (9), (10), and (11) into (7) we get

$$\zeta = -\frac{2}{\sqrt{\beta}}\int_0^w \frac{1}{sech^2 w \cdot \tanh w} \cdot \frac{\sinh w}{\cosh^3 w}\, dw = -\frac{2}{\sqrt{\beta}}\int_0^w \frac{\cosh^2 w \cdot \cosh w}{\sinh w} \cdot \frac{\sinh w}{\cosh^3 w}\, dw$$

$$\Rightarrow \quad \zeta = -\frac{2}{\sqrt{\beta}}\int_0^w dw = -\frac{2}{\sqrt{\beta}}\, w$$

With (8) the transform back to ζ is done and we obtain:

$$\xi = -\frac{2}{\sqrt{\beta}}\, sech^{-1}\sqrt{\frac{2z}{\beta}} \quad \Rightarrow \quad z(\xi) = \frac{\beta}{2}\, sech^2\left(\frac{\sqrt{\beta}}{2}\,\xi\right)$$

Now we use (2) and we finally get

(12) $$u(x,t) = \frac{\beta}{2}\, sech^2\left[\frac{\sqrt{\beta}}{2}(x - \beta t)\right]$$

Remarks:
In order to have a real solution the quantity β must be a positive number. As it is easily seen from (12) for β > 0 the solitary wave moves to the right. The second point is that the amplitude is proportional to the speed which is indicated by the value of β. Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves.
To perform superpositions later (see §4) we consider the following: If - instead of (8) - we select the transformation

(13) $$s := -\tfrac{1}{2}\beta\, csch^2 w$$

then in the same way as we obtained the solution (12) we will obtain another solution which is:

$$u(x,t) = -\frac{\beta}{2} csch^2\left[\frac{\sqrt{\beta}}{2}(x-\beta t)\right] \tag{14}$$

The solution (14) is an irregular solution to the KdV Equation. It has a singularity for vanishing argument of the cosech-function, i.e. for the line in the *x*-*t*-plane with

$$x - \beta t = 0 \;\; \Leftrightarrow \;\; t = \frac{1}{\beta} x$$

The proof that both functions (12) and (14) are solutions to the original KdV Equation can be easily varified with the help of *Mathematica* as shown in the sequel

```
PDEKdV[u_, x_, t_] :=
 ∂t u[x, t] + 6 u[x, t] ∂x u[x, t] + ∂{x,3} u[x, t] == 0
```

$$\texttt{sol1[x, t] :=} \; \frac{\beta}{2}\,\texttt{Sech}\left[\frac{\sqrt{\beta}\,(x-\beta\, t)}{2}\right]^2$$

$$\texttt{sol2[x, t] :=} \; -\frac{\beta}{2}\,\texttt{Csch}\left[\frac{\sqrt{\beta}\,(x-\beta\, t)}{2}\right]^2$$

Next these solutions **`sol1`** and **`sol2`** are inserted into **`PDEKdV`**:

```
Simplify[PDEKdV[sol1, x, t]]

True

Simplify[PDEKdV[sol2, x, t]]

True
```

§ 3: Exact Solution with Bäcklund transfom

We are now going to construct a solution to (1) with the help of Bäcklund transform since this technique is used later for non-linear superposition. We introduce a function v with the property $v_x = u$ and it follows from (1):

$$v_{tx} + 6 v_x v_{xx} + v_{xxxx} = \frac{\partial}{\partial}\left[v_t + 3 v_x^2 + v_{xxx}\right] = 0 \tag{15}$$

Integration with respect to x yields

$$v_t + 3 v_x^2 + v_{xxx} = f(t) \tag{16}$$

where the function on the right hand side is the integration function.
The new function v^* is introduced with

$$v^* = v - \int^t f(s)\,ds$$

The solution u to (1) will be obtained from the solution v to (16) and since $v_x^* = v_x$ there is no need for a distinction between v^* and v. So we may set f ≡ 0 without loss of generality.
Now our purpose is to solve the PDE

$$v_t + 3v_x^2 + v_{xxx} = 0 \tag{17}$$

A Bäcklund transform which leaves an equation invariant is called *Auto-Bäcklund transform*. This Auto-Bäcklund transform is given by the following set of equations:

$$\begin{aligned}
&\tilde{v}_\xi = -v_x + \beta - \tfrac{1}{2}(v-\tilde{v})^2 \\
&\tilde{v}_\tau = -v_t + (v-\tilde{v})(v_{xx}-\tilde{v}_{\xi\xi}) - 2(v_x^2 + v_x\tilde{v}_\xi + \tilde{v}_\xi^2) \\
&\xi = x; \qquad \tau = t
\end{aligned} \tag{18}$$

The parameter β is the so called Bäcklund parameter, for details we refer to Vvedenskii (1992), chapter 9.
We now may generate a non trivial solution by applying (18) to the trivial solution (which clearly fulfills (17)). With $\tilde{v} \equiv 0$ the first two lines of (18) result into

$$\begin{aligned}
v_x &= \beta - \tfrac{1}{2}v^2 \\
v_t &= v\,v_{xx} - 2v_x^2
\end{aligned} \tag{19}$$

The first formula in (19) may easily be integrated and be inserted into the second. Two types of solutions fulfill (19):

$$\begin{aligned}
v(x,t) &= \sqrt{2\beta}\,\tanh\left[\sqrt{\tfrac{\beta}{2}}\,(x-2\beta t)\right] \\
\bar{v}(x,t) &= \sqrt{2\beta}\,\coth\left[\sqrt{\tfrac{\beta}{2}}\,(x-2\beta t)\right]
\end{aligned} \tag{20}$$

It is easy to verify that both functions (20) do fulfill (19) and (17). The second function has a singularity for vanishing argument of coth. This irregular solution here and in future is denoted with a bar. Now it is easy by integrating with respect to x to deduce from (20) two solutions (a regular one and an irregular one) to the KdV equation (1). These solutions are:

(21)
$$u(x,t) = v_x(x,t) = \beta\ sech^2\left[\sqrt{\frac{\beta}{2}}\,(x-2\,\beta\,t)\right]$$
$$\bar{u}(x,t) = \bar{v}_x(x,t) = -\beta\ csch^2\left[\sqrt{\frac{\beta}{2}}\,(x-2\,\beta\,t)\right]$$

Again we are led to the same solution we already found with (12) and (14). The only difference is the parameter. If we denote the parameter β in (21) by β^* and compare it with the parameter β from (12) and (14) then the relation $\beta^* = \frac{\beta}{2}$ holds. We may verify the steps with *Mathematica*:

```
PDEori[v_, x_, t_] :=
  ∂t v[x, t] + 3 (∂x v[x, t])^2 + ∂{x,3} v[x, t] == 0

PDE1[v_, x_, t_] := ∂x v[x, t] == β - 1/2 v[x, t]^2

PDE2[v_, x_, t_] :=
  ∂t v[x, t] == v[x, t] ∂{x,2} v[x, t] - 2 (∂x v[x, t])^2
```

With **DSolve** we may obtain a solution with the following delayed function:

```
solu[x_, t_] := v[x, t] /.
  Simplify[DSolve[PDE1[v, x, t], v[x, t], {x, t}]][[1]]

solu[x, t]
```

$$\sqrt{2}\,\sqrt{\beta}\ \mathrm{Tanh}\left[\frac{\sqrt{\beta}\,(x-2\,C[1][t])}{\sqrt{2}}\right]$$

Typically the arbitrary function is denoted by **C[1][t]**. Insertion into **PDE1** is done and the attempt is made whether the Coth-function instead of the Tanh-function fulfills **PDE1**:

```
Simplify[PDE1[solu, x, t]]

True

Simplify[PDE1[solu /. Tanh → Coth, x, t]]

True
```

Both the functions fulfill (19), first equation. Now the second equation is solved by:

```
spec = C[1][t] /. DSolve[PDE2[solu, x, t], C[1][t], t][[1]]

t β + C[1]

spec0 = spec /. C[1] → 0

t β
```

where the integration constant has been set to zero. With this result we define the two functions:

```
soluspec1[x_, t_] :=
  solu[x, t] /. {∂t C[1][t] → ∂t spec0 , C[1][t] → spec0}
```

```
soluspec2[x_, t_] := soluspec1[x, t] /. Tanh → Coth
```

```
soluspec1[x, t]
```

$$\sqrt{2}\,\sqrt{\beta}\,\text{Tanh}\left[\frac{\sqrt{\beta}\,(x-2\,t\,\beta)}{\sqrt{2}}\right]$$

```
soluspec2[x, t]
```

$$\sqrt{2}\,\sqrt{\beta}\,\text{Coth}\left[\frac{\sqrt{\beta}\,(x-2\,t\,\beta)}{\sqrt{2}}\right]$$

Both these functions clearly fulfill the second equation of (19) and the PDE (17):

```
Simplify[PDE2[soluspec1, x, t]]
```

```
True
```

```
Simplify[PDE2[soluspec2, x, t]]
```

```
True
```

```
Simplify[PDEori[soluspec1, x, t]]
```

```
True
```

```
Simplify[PDEori[soluspec2, x, t]]
```

```
True
```

Finally the solution to the KdV equation is given by the first derivative with respect to *x*:

```
∂x soluspec1[x, t]
```

$$\beta\,\text{Sech}\left[\frac{\sqrt{\beta}\,(x-2\,t\,\beta)}{\sqrt{2}}\right]^2$$

```
∂x soluspec2[x, t]
```

$$-\beta\,\text{Csch}\left[\frac{\sqrt{\beta}\,(x-2\,t\,\beta)}{\sqrt{2}}\right]^2$$

Though the irregular solution is not acceptable in physical sense it will be used to construct higher order solutions by using the non-linear superposition principle.

§ 4: Non-linear superposition

Let Φ be any solution to (17) and let $\Phi(\beta)$ be a solution obtained from Φ by applying the Bäcklund transform with the Bäcklund parameter β. To remind what we did: We had chosen the trivial solution $\Phi \equiv 0$ and from that we received (20) as a solution to (17).
We now consider two solutions $\Phi(\beta_1)$ and $\Phi(\beta_2)$ obtained by applying the Bäcklund transform to Φ with parameters β_1 and β_2.
From (18) we have (only the first line is used):

$$\begin{aligned} \Phi_x(\beta_1) &= -\Phi_x + \beta_1 - \tfrac{1}{2}(\Phi - \Phi(\beta_1))^2 \\ \Phi_x(\beta_2) &= -\Phi_x + \beta_2 - \tfrac{1}{2}(\Phi - \Phi(\beta_2))^2 \end{aligned} \tag{22}$$

Let $\Phi(\beta_2,\beta_1)$ be the solution obtained from Φ by successive application of (18), first line - first step with parameter β_1 and second step with parameter β_2. Then we have

$$\Phi_x(\beta_2,\beta_1) = -\Phi_x(\beta_1) + \beta_2 - \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2,\beta_1))^2 \tag{23}$$

In the same way let $\Phi(\beta_1,\beta_2)$ be the solution by first applying β_2 and afterwards β_1.
Instead of (23) we then have

$$\Phi_x(\beta_1,\beta_2) = -\Phi_x(\beta_2) + \beta_1 - \tfrac{1}{2}(\Phi(\beta_2) - \Phi(\beta_1,\beta_2))^2 \tag{24}$$

We now demand that

$$\Phi(\beta_2,\beta_1) = \Phi(\beta_1,\beta_2) =: \Psi \tag{25}$$

and try to solve (23) and (24) for Ψ. Subtraction of the second line in (22) from the first yields:

$$\Phi_x(\beta_1) - \Phi_x(\beta_2) = \beta_1 - \beta_2 - \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2))\cdot(\Phi(\beta_1) + \Phi(\beta_2) - 2\,\Phi) \tag{26}$$

Subtracting (24) from (23) end using (25) leads to

$$\Phi_x(\beta_1) - \Phi_x(\beta_2) = \beta_2 - \beta_1 - \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2))\cdot(\Phi(\beta_1) + \Phi(\beta_2) - 2\,\Psi) \tag{27}$$

By eliminating the left-hand sides in (26) and (27) we get:

$$\beta_1 - \beta_2 - \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2)) \cdot (\Phi(\beta_1) + \Phi(\beta_2) - 2\Phi)$$

$$= \beta_2 - \beta_1 - \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2)) \cdot (\Phi(\beta_1) + \Phi(\beta_2) - 2\Psi)$$

$$\Rightarrow \quad 2(\beta_1 - \beta_2) = (\Psi - \Phi)(\Phi(\beta_1) - \Phi(\beta_2))$$

$$\Rightarrow \quad \Psi = \Phi + 2\,\frac{\beta_1 - \beta_2}{\Phi(\beta_1) - \Phi(\beta_2)} \tag{28}$$

We have to show now that the function Ψ from (28) indeed is a solution to (17):
The expression (28) is differentiated with respect to x and to t and three times with respect to x and is inserted into (17). The result is:

$$\begin{aligned}
\Psi_t + 3\Psi_x^2 + \Psi_{xxx} &= \Phi_t + \Phi_{xxx} \\
&-2(\beta_1 - \beta_2)\frac{\Phi_t(\beta_1) - \Phi_t(\beta_2)}{(\Phi(\beta_1) - \Phi(\beta_2))^2} + 3\left[\Phi_x - 2(\beta_1 - \beta_2)\frac{\Phi_x(\beta_1) - \Phi_x(\beta_2)}{(\Phi(\beta_1) - \Phi(\beta_2))^2}\right]^2 \\
&-2(\beta_1 - \beta_2)\frac{\Phi_{xxx}(\beta_1) - \Phi_{xxx}(\beta_2)}{(\Phi(\beta_1) - \Phi(\beta_2))^2} - 12(\beta_1 - \beta_2)\frac{(\Phi_x(\beta_1) - \Phi_x(\beta_2))^3}{(\Phi(\beta_1) - \Phi(\beta_2))^4} \\
&+ 12(\beta_1 - \beta_2)\frac{(\Phi_x(\beta_1) - \Phi_x(\beta_2)) \cdot (\Phi_{xx}(\beta_1) - \Phi_{xx}(\beta_2))}{(\Phi(\beta_1) - \Phi(\beta_2))^3}
\end{aligned} \tag{29}$$

The second derivates with respect to x in (26) are formed resulting in:

$$\begin{aligned}
\Phi_{xx}(\beta_1) - \Phi_{xx}(\beta_2) &= \tfrac{1}{2}(\Phi_x(\beta_1) - \Phi_x(\beta_2)) \cdot (\Phi(\beta_1) + \Phi(\beta_2) - 2\Phi) \\
&- \tfrac{1}{2}(\Phi(\beta_1) - \Phi(\beta_2)) \cdot (\Phi_x(\beta_1) + \Phi_x(\beta_2) - 2\Phi_x)
\end{aligned} \tag{30}$$

The formula (30) has to be inserted into the last summand in (29) thus eliminating the second derivatives. If we do so and keep in mind that each of the functions Φ, $\Phi(\beta_1)$, $\Phi(\beta_2)$ are solutions to (17) then we get from (29):

$$\begin{aligned}
\Psi_t + 3\Psi_x^2 + \Psi_{xxx} &= 12(\beta_1 - \beta_2)^2\frac{(\Phi_x(\beta_1) - \Phi_x(\beta_2))^2}{(\Phi(\beta_1) - \Phi(\beta_2))^4} - 12(\beta_1 - \beta_2)\frac{(\Phi_x(\beta_1) - \Phi_x(\beta_2))^3}{(\Phi(\beta_1) - \Phi(\beta_2))^4} \\
&+ 6(\beta_1 - \beta_2)\frac{(\Phi_x(\beta_1) - \Phi_x(\beta_2))^2(\Phi(\beta_1) + \Phi(\beta_2) - 2\Phi)}{(\Phi(\beta_1) - \Phi(\beta_2))^3}
\end{aligned} \tag{31}$$

Now from (26) the expression $\Phi(\beta_1) + \Phi(\beta_2) - 2\,\Phi$ is calculated and inserted into the last term of (31). After some rearrangement the sum of the terms on the right-hand-side of (31) is zero thus showing that the function Ψ is indeed a solution to (17).

The main formula for constructing further solutions is (28). If we take $\Phi \equiv 0$ which clearly is a solution to (17) and if we take the regular and the irregular solutions (20) which fulfill (17), then these two solutions play the role of $\Phi(\beta_1)$ and $\Phi(\beta_2)$ in (28).
Using two regular solutions (first solution of (20) with two different values of β) would lead to an irregular solution Ψ by using (28). This can be seen as follows from (20), first equation, with a fixed value for the parameter β:

$$\lim_{(x-2\beta t)\to\pm\infty} v(x,t) = \lim_{(x-2\beta t)\to\pm\infty} \sqrt{2\beta}\,\tanh\left[\sqrt{\frac{\beta}{2}}\,(x-2\,\beta\,t)\right] = \pm\sqrt{2\beta} \tag{32}$$

Since the function v is continous there will always be one value of the argument for wich the denominator in (28) is zero, leading to a singularity in Ψ.
So we use both, the regular and the irregular solution from (20) to construct a regular solution by using (28). The solutions from (20) are denoted by $v(\beta_1)$ and $\bar{v}(\beta_2)$ analogous to what we did here before, the bar again denotes the irregular solution. The two Bäcklund parameters are chosen as $\beta_2 > \beta_1$. This ensures that the denominator in (28) does not vanish for any values of the arguments. Thus we get a superpositioned function:

$$\Psi(x,t) = \frac{2\,(\beta_1 - \beta_2)}{v(\beta_1,x,t) - \bar{v}(\beta_2,x,t)}; \quad \beta_2 > \beta_1 \tag{33}$$

Since the function Ψ from (33) is a solution to (17) it has to be integrated with respect to x to obtain a superpositioned solution to the original KdV equation.
This process of supositioning a regular and an irregular solution to (17) can be iterated. We have to apply (28) again to 3 solutions. Instead of Φ in (28) we now take the regular solution from (20) with a Bäcklund parameter β which is different from β_1 and β_2 and name it $v(\beta,x,t)$. The role of the function $\Phi(\beta_1)$ in (28) is now played by the function Ψ from (33) for the two Bäcklund parameters β_1 and β and thus is named as $\Phi(\beta_1,\beta,x,t)$.

$$\Phi(\beta_1,\beta,x,t) = \frac{2\,(\beta - \beta_1)}{v(\beta,x,t) - \bar{v}(\beta_1,x,t)}; \quad \beta_1 > \beta \tag{34}$$

This solution is regular, a corresponding irregular solution (denoted by a bar) then is gained by combining two regular solutions as mentioned above:

$$\bar{\Phi}(\beta_2,\beta,x,t) = \frac{2\,(\beta_2 - \beta)}{v(\beta_2,x,t) - v(\beta,x,t)} \tag{35}$$

The two functions of the denominators of (34) and (35) are again the solutions (20).
If we combine the two functions of (34) and (35) we want to get a regular solution for a 3-soliton solution. It is necessary then that it holds $\beta_2 - \beta > \beta_1 - \beta$. Together with the regularity condition in (34)/right we then have $\beta_2 > \beta_1 > \beta$ as a regularity condition.
Finally according to (28) the new 3-soliton solution is

$$\Psi(x,t) = v(\beta,x,t) + \frac{2(\beta_1 - \beta_2)}{\Phi(\beta_1,\beta,x,t) - \bar{\Phi}(\beta_2,\beta,x,t)} \tag{36}$$

Again (36) is a solution to (17), it has to be intgrated with respect to *x* to get a solution to the original KdV equation.
Principally by superposition a regular *n*-soliton can be constructed from a pair of irregular and regular (*n*-1)-solutions.

§ 5: *Mathematica* code for one, two and three solitons

The solutions from (21) as well as the solutions got from (33) and (36) are coded in *Mathematica* as follows:

```
KdV[Bval_, x_, t_] := Block[{vv, v, w, superpos2, superpos3,
   ϕIrr, ϕReg, Solution},
  Clear[x, t];
  v[xx_, tt_, β_] := Sqrt[2 β] Tanh[Sqrt[β/2] (xx - 2 β tt)];
  w[xx_, tt_, β_] := Sqrt[2 β] Coth[Sqrt[β/2] (xx - 2 β tt)];
  superpos2[xx_, tt_, β1_, β2_] := (2 (β1 - β2))/(v[xx, tt, β1] - w[xx, tt, β2]);
  ϕIrr[xx_, tt_, β2_, β_] := (2 (β2 - β))/(v[xx, tt, β2] - v[xx, tt, β]);
  ϕReg[xx_, tt_, β1_, β_] := (2 (β - β1))/(v[xx, tt, β] - w[xx, tt, β1]);
  superpos3[xx_, tt_, β_, β1_, β2_] :=
   v[xx, tt, β] + (2 (β1 - β2))/(ϕReg[xx, tt, β1, β] - ϕIrr[xx, tt, β2, β]);
  vv = Sort[Bval];
  If[Length[Bval] == 1, Solution = ∂x v[x, t, β] /. β → vv[[1]],
   If[Length[Bval] == 2, Solution = ∂x superpos2[x, t, β1, β2] /.
      {β1 → vv[[1]], β2 → vv[[2]]},
    Solution = ∂x superpos3[x, t, β, β1, β2] /.
      {β → vv[[1]], β1 → vv[[2]], β2 → vv[[3]]}]]]
```

The first parameter of this block is a list. If it contains one element a one-soliton solution is calculated, if it contains two or three elements 2- or 3-soliton solutions are calculated. It is ensured in the code that the condition $\beta_2 > \beta_1 > \beta$ is not violated.
Calls, results, and proofs are:

KdVOp[u_] := ∂t u + 6 u ∂x u + ∂{x,3} u

One = KdV[{β1}, x, t]

$$\text{Sech}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]^2 \beta_1$$

Simplify[KdVOp[One]]

0

Two = KdV[{β1, β2}, x, t]

$$-\frac{2\,(\beta_1-\beta_2)\left(\text{Sech}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]^2 \beta_1+\text{Csch}\left[\frac{\sqrt{\beta_2}\,(x-2\,t\,\beta_2)}{\sqrt{2}}\right]^2 \beta_2\right)}{\left(-\sqrt{2}\,\text{Coth}\left[\frac{\sqrt{\beta_2}\,(x-2\,t\,\beta_2)}{\sqrt{2}}\right]\sqrt{\beta_2}+\sqrt{2}\,\sqrt{\beta_1}\,\text{Tanh}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]\right)^2}$$

Simplify[KdVOp[Two]]

0

For three solitons an analytical solution was obtained,but *Mathematica* was not able to verify that it truely fulfills the KdV equation:

Three = KdV[{β1, β2, β3}, x, t]

$$\text{Sech}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]^2 \beta_1-\Bigg(2\,(\beta_2-\beta_3)$$

$$\left(-\frac{2\,(\beta_1-\beta_2)\left(\text{Sech}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]^2 \beta_1+\text{Csch}\left[\frac{\sqrt{\beta_2}\,(x-2\,t\,\beta_2)}{\sqrt{2}}\right]^2 \beta_2\right)}{\left(-\sqrt{2}\,\text{Coth}\left[\frac{\sqrt{\beta_2}\,(x-2\,t\,\beta_2)}{\sqrt{2}}\right]\sqrt{\beta_2}+\sqrt{2}\,\sqrt{\beta_1}\,\text{Tanh}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]\right)^2}+\right.$$

$$\left.\frac{2\,(-\beta_1+\beta_3)\left(-\text{Sech}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]^2 \beta_1+\text{Sech}\left[\frac{\sqrt{\beta_3}\,(x-2\,t\,\beta_3)}{\sqrt{2}}\right]^2 \beta_3\right)}{\left(-\sqrt{2}\,\sqrt{\beta_1}\,\text{Tanh}\left[\frac{\sqrt{\beta_1}\,(x-2\,t\,\beta_1)}{\sqrt{2}}\right]+\sqrt{2}\,\sqrt{\beta_3}\,\text{Tanh}\left[\frac{\sqrt{\beta_3}\,(x-2\,t\,\beta_3)}{\sqrt{2}}\right]\right)^2}\right)\Bigg) \Big/$$

$$\left(\frac{2\,(\beta_1-\beta_2)}{-\sqrt{2}\ \mathrm{Coth}\left[\frac{\sqrt{\beta_2}\ (x-2\,t\,\beta_2)}{\sqrt{2}}\right]\sqrt{\beta_2}+\sqrt{2}\ \sqrt{\beta_1}\ \mathrm{Tanh}\left[\frac{\sqrt{\beta_1}\ (x-2\,t\,\beta_1)}{\sqrt{2}}\right]}-\frac{2\,(-\beta_1+\beta_3)}{-\sqrt{2}\ \sqrt{\beta_1}\ \mathrm{Tanh}\left[\frac{\sqrt{\beta_1}\ (x-2\,t\,\beta_1)}{\sqrt{2}}\right]+\sqrt{2}\ \sqrt{\beta_3}\ \mathrm{Tanh}\left[\frac{\sqrt{\beta_3}\ (x-2\,t\,\beta_3)}{\sqrt{2}}\right]}\right)\text{^}2$$

It is easy now to write 3 additional blocks in Mathematica to produce a 3D plot, a contour plot, and a sequence of 2D plots which can be animated. The code is straightforward, the results are not shown here but may be seen visiting Brauer (2000), further information is got by visiting Eilbeck (1998).

These analytical solutions are very helpful when applying numerical methods since a comparision is possible, see de Frutos & Sanz-Serna (1997) and Schiesser (1994). A different approach in obtaining solutions to the KdV equation is shown by Varley & Seymour (1998).

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The obtained solutions to the modied Korteweg-de Vries equation can simply be categorized by two types: solitons and breathers, together with their limit cases. Besides, we give rational solutions to the mod-ied Korteweg-de Vries equation in Wromskian form. This is derived with the help of the Galilean transformed modied Korteweg-de Vries equation. Finally, typical dynamics of the obtained solutions is analyzed and illustrated. We list out the obtained solutions and their corresponding basic Wronskian vectors in the conclusion part.Â With regard to exact solutions, many classical solving methods, such as Hirotaâ€™s bilinear. method[17], the IST[18, 19], commutation methods[20] and Wronskian technique[21, 22, 23] has. been used to solve the mKdV equation.