Vector and Geometric Calculus

Fifth printing, corrected and slightly revised

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Geometry without algebra is dumb! - Algebra without geometry is blind!

- David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.

- Allan McRobie and Joan Lasenby

To Ellen

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Preface

This text, *Vector and Geometric Calculus*, is intended for the second year vector calculus course. It is a sequel to my text *Linear and Geometric Algebra.* That text is a prerequisite for this one.

Linear algebra and vector calculus have provided the basic vocabulary of mathematics in dimensions greater than one for the past one hundred years. Geometric algebra generalizes linear algebra in powerful ways. Similarly, geometric calculus generalizes vector calculus in powerful ways.

Traditional vector calculus topics are covered here, as they must be, since readers will encounter them in other texts and out in the world.

The final chapter is an introduction to differential geometry, used today in many disciplines, including architecture, computer graphics, computer vision, econometrics, engineering, geology, image processing, and physics.

Large parts of vector calculus are confined to $\mathbb{R}^3$ due to the extensive use of the cross product. Tensors and differential forms are two traditional formalisms used to extend to higher dimensions. Geometric calculus provides an at once simpler and more powerful way to break loose from $\mathbb{R}^3$.[1] Appendix C provides a short comparison of differential forms and geometric calculus.

Linear algebra is the natural language in which to express vector calculus. Yet even today it is unusual for a vector calculus text to have a linear algebra prerequisite. This has to do, I suppose, with publishers insisting that authors write to the largest possible audience. I use linear and geometric algebra ideas freely and pervasively to advantage.

Linear and geometric algebra and also differential vector and geometric calculus (Part II of this book) are excellent places to help students better understand and appreciate rigor. But for integral calculus (Part III) rigorous proofs at the level of this book are mostly impossible. So I do not try.

Instead, I use the language of infinitesimals, while making it clear that they do not exist within the real number system. I believe that the first and most important way to understand integrals is intuitively: they "add infinitely many infinitesimal parts to give a whole".

[1] D. Hestenes and G. Sobczyk have argued the superiority of geometric calculus over differential forms (*Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht Holland 1984, Section 6.4.)

Others endorse this approach: "An approach based on [infinitesimals] closely reflects the way most scientists and engineers successfully use calculus."[2] From Lagrange: "When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results, ... we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.[3] And even Cauchy: "My main aim has been to *reconcile* the rigor which I have made a law in my Cours d'Analyse, with the simplicity that comes from the direct consideration of infinitely small quantities"[4] (Emphasis added.)

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that students attempt them while reading the text. That way they immediately confront the concept and get feedback on their understanding. There are also more challenging problems at the end of most sections – almost 200 in all.

The exercises replace the "worked examples" common in most mathematical texts, which serve as "templates" for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Some exercises and problems require the use of the free multiplatform Python module $\mathcal{GA}$lgebra. It is based on the Python symbolic computer algebra library SymPy (Symbolic Python). The file $\mathcal{GA}$lgebra Primer describes the installation and use of the module. It is available at the book's web site and is bundled with the $\mathcal{GA}$lgebra distribution.[5]

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file "VAGC Instructor.pdf" at the book's web site. Take it for what it is worth.

The first part of the index is a symbol index.

Some material which is difficult or less important is printed in this smaller font.

There are several appendices. Appendix A reviews some parts of *Linear and Geometric Algebra* used in this book. Appendix B provides a list of some geometric calculus formulas from the book. Appendix C provides a short comparison of differential forms and geometric calculus. Appendix D provides some technical results.

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[2] Tevian Dray and Corinne Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, The College Mathematics Journal **34**, 283-290 (2003).

[3] Méchanique Analytique, Preface; Ouveres, t. 2 (Paris, 1988), p. 14.

[4] Quoted in *Cauchy's Continuum*, Karin Katz and Mikhail Katz, Perspectives on Science **19**, 426-452. Also at arXiv:1108.4201v2.

[5] https://github.com/brombo/galgebra

Numbered references to theorems, figures, etc. preceded by "LAGA" are to *Linear and Geometric Algebra.*

There are several URL's in the text. To save you typing, I have put them in a file "URLs.txt" at the book's web site.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book's web site as appropriate.

Acknowledgements. I thank Dr. Eric Chisolm, Greg Grunberg, Professor Philip Kuntz, James Murphy, and Professor John Synowiec for reading all/most of the text and providing and helpful comments and advice. Professor Mike Taylor answered several questions. I give special thanks to Greg Grunberg and James Murphy. Grunberg spotted many errors, made many valuable suggestions and is an eagle eyed proofreader. Murphy suggested major revisions in the ordering of my chapters.

I also thank the ever cooperative Alan Bromborsky for extending $\mathcal{GA}$lgebra to make it more useful to the readers of this book.

Thanks again to Professor Kate Martinson for help with the cover design.

> In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, ... Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius.
>
> - C. F. Gauss

Second Printing

This second printing has no major changes. It corrects all errors known to me in the first printing. There are many improvements in presentation and a small amount of new material. The numbering of equations, theorems, etc. is unchanged from the first printing. Exceptions: I have exchanged Problems 7.4.8 and 7.4.7. And Problem 10.3.4 is now Problem 10.1.3.

I thank again Gregory Grunberg for many suggestions and expert proofreading. Christoph Bader and Dr. Gavin Polhemous pointed out shortcomings in the notation of Section 5.4. And I thank Dr. Manuel Reenders, a recent arrival, for many suggestions and corrections, especially with regard to the exercises and problems.

Third Printing

This third printing has no major changes. It corrects all errors known to me in the second printing. There are many improvements in presentation and a small amount of new material. The numbering of equations, theorems, etc. is unchanged from the second printing. Exception: The sections of Chapter 3 have been reordered to avoid forward references. As a consequence, the numberings have changed. Chapter B, *Software*, describes the latest version of Alan Bromborsky's SymPy module $\mathcal{GA}$lgebra (renamed from *GA*).

I thank a new eagle eyed reader, Nicholas H. Okamoto, for sending me errata.

Fourth Printing

This fourth printing has no major changes. It corrects all errors known to me in the third printing. There are many improvements in presentation and a small amount of new material. The numbering of equations, theorems, etc. is unchanged from the third printing. However, page numbers have changed, as I increased the font size to match that of *Linear and Geometric Algebra*.

I have removed the old Appendix B, *Software*, which documented Alan Bromborsky's Python module $\mathcal{GA}$lgebra. The documentation there became out of date as the module improved. Current documentation is available at this book's web site in the file $\mathcal{GA}$lgebraPrimer.pdf. It is also bundled with the $\mathcal{GA}$lgebra distribution.

Fifth Printing

This fifth printing has no major changes. It corrects all errors known to me in the fourth printing. There are many improvements in presentation and a small amount of new material. Some material in Section 11.2 has been moved to Section 5.5 and expanded there, causing some changes in the numbering of equations, theorems, etc. in those sections.

Page numbers have changed due to all this.

$\mathcal{GA}$lgebra is making a transition from Python 2 and Jupyter to Python 3 and Jupyter Lab. See $\mathcal{GA}$lgebra's GitHub page.[6]

I thank the very careful new reader Professor Mark R. Treuden for helpful comments and corrections.

[6] https://github.com/brombo/galgebra

To the Student

Appendix A is a review of some items from *Linear and Geometric Algebra* (LAGA) used in this book. A quick read through it might be helpful before starting this book.

I repeat here my advice from *Linear and Geometric Algebra.*

Research clearly shows that *actively* engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

- ~~Read~~ Study the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you will understand.

- Instructors in your previous mathematics courses have probably urged you to try to *understand*, rather than simply memorize. That advice is especially appropriate for this text.

- Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand while you read.

- Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.

- Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.

- Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!

The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that "mathematical frame of mind".

Write your solutions neatly in clear correct English.

- Proofs are important, but perhaps less so than the above. On a first reading, don't get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.

- Take the above points seriously!

The World Wide Web makes it possible for me to leave out material that I would otherwise have to include. For example, the book refers to the *Coulomb force* without defining it. Perhaps you already know what it is. If not, and you want to know, actively engage the course material: Google it.

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Buy Vector and Geometric Calculus on Amazon.com âœ“ FREE SHIPPING on qualified orders.Â Vector and Geometric Calc has been added to your Cart. Add to Cart. Turn on 1-Click ordering for this browser. In mathematics, geometric calculus extends the geometric algebra to include differentiation and integration. The formalism is powerful and can be shown to encompass other mathematical theories including differential geometry and differential forms.[1].Â is a vector tangent to the manifold, then indeed both the geometric derivative and intrinsic derivative give the same directional derivative: a â‹… âˆ‚ F = a â‹… âˆ‡ F . {\displaystyle a\cdot \partial F=a\cdot \nabla F.} Although this operation is perfectly valid, it is not always useful because.